Factorization in Finitely-Presented Monoids

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Monoid Terminology

Notation and General Terminology

- M denotes a multiplicative monoid throughout.
- $M^\times$ denotes the group of invertible elements of M.
- M is *reduced* if $M^\times = \{1\}$.
- M is *cancellative* if each of $ab = ac$ and $ba = ca$ implies that $b = c$, for all $a, b, c \in M$.
- M is *unit-cancellative* if each of $a = ab$ and $a = ba$ implies that $b \in M^\times$, for all $a, b, c \in M$.

Atoms

- $a \in M$ is an *atom* if $a \notin M^\times$, and $a = bc$ implies that $b \in M^\times$ or $c \in M^\times$, for all $b, c \in M$.
- $\mathcal{A}(M)$ denotes the set of atoms of M.
- M is *atomic* if every element of $M \setminus M^\times$ is a product of atoms.

Some Traditional Factorization Terminology

Length Sets

- $\mathsf{L}_M(a) := \{k \in \mathbb{N}^+ \mid \exists u_1, \ldots, u_k \in \mathcal{A}(M)\ (a = u_1 \ldots u_k)\}$ is the *length set of* $a \in M$.
- $\mathcal{L}(M) := \{\mathsf{L}_M(a) \mid a \in M\}$ is the *system of length sets of* M.

Distances

- $[n, m] := \{l \in \mathbb{Z} \mid n \leq l \leq m\}$ for all $n, m \in \mathbb{Z}$
- $d \in \mathbb{N}^+$ is a *distance of* $L \in \mathcal{L}(M)$ if there exists $k \in L$ such that $[k, k+d] \cap L = \{k, k+d\}$.
- $\Delta(L)$ denotes the set of all distances of $L \in \mathcal{L}(M)$.
- $\Delta(M) := \bigcup_{L \in \mathcal{L}(M)} \Delta(L)$.

Unions and Elasticities

- $\mathcal{U}_k(M) := \bigcup \{L \in \mathcal{L}(M) \mid k \in L\}$ is the *union of length sets* containing $k \in \mathbb{N}$.
- $\rho_k(M) := \sup(\mathcal{U}_k(M)) \in \mathbb{N} \cup \{\infty\}$ is the *kth elasticity of* M, for all $k \in \mathbb{N}$ (provided that $\mathcal{U}_k(M) \neq \emptyset$).

Structure Theorem for Unions

M satisfies the *Structure Theorem for Unions* if there exist $k^* \in \mathbb{N}^+$ and $m \in \mathbb{N}$, such that for all $k \geq k^*$,

$$\mathcal{U}_k(M) = n_k + \left(L'_k \cup L^*_k \cup L''_k\right) \subseteq n_k + r\mathbb{Z},$$

where

- $r = \min(\Delta(M))$,
- $n_k \in \mathbb{Z}$,
- $L'_k \subseteq [-m, -1]$,
- L^*_k is a nonempty arithmetic progression with difference r and $\min(L^*_k) = 0$,
- $L''_k \subseteq \sup(L^*_k) + [1, m]$ if L^*_k is finite, and $L''_k = \emptyset$ if L^*_k is infinite.

Theorem (Geroldinger, 2016)

Suppose that M is commutative and atomic, with $\Delta(M)$ finite but nonempty. Suppose further that either $\rho_n(M) = \infty$ for some $n \in \mathbb{N}^+$, or there exists $l \in \mathbb{N}^+$ such that $\rho_k(M) - \rho_{k-1}(M) \leq l$ for all $k \geq 2$. Then M satisfies the Structure Theorem for Unions.

Initial Questions

Question

Must a (noncommutative) finitely-presented, atomic, unit-cancellative monoid M have finite elasticities $\rho_k(M) = \sup(\bigcup\{L \in \mathcal{L}(M) \mid k \in L\})$?

Question

For a (noncommutative) finitely-presented, atomic, unit-cancellative monoid M with finite elasticities $\rho_k(M)$, must it be the case that the differences $\rho_k(M) - \rho_{k-1}(M)$ are bounded?

Question

Must a (noncommutative) finitely-presented, atomic, unit-cancellative monoid satisfy the Structure Theorem for Unions?

Example

Question

Must a finitely-presented, atomic, unit-cancellative monoid M have finite elasticities $\rho_k(M)$?

- Let $M := \langle w, x, y, z \mid (xy, wxyz) \rangle$.
- *Claim 1:* $\mathcal{A}(M) = \{w, x, y, z\}$ *and* M *is atomic.*
- *Claim 2:* M *is cancellative and reduced.*
- *Claim 3:* $\rho_k(M)$ *is infinite for each* $k \geq 2$.

  Let $k \geq 2$. Then

$$x^{k-2}xy = x^{k-2}wxyz = x^{k-2}w^2xyz^2 = \cdots = x^{k-2}w^nxyz^n$$

  for all $n \geq 1$. So

$$\{k, k+2, k+4, \dots\} \subseteq L_M(x^{k-2}xy) \subseteq \mathcal{U}_k(M) = \bigcup\{L \in \mathcal{L}(M) \mid k \in L\},$$

  and hence

$$\rho_k(M) = \sup(\mathcal{U}_k(M)) = \infty.$$

Generator Factorization Terminology

Presented Monoids

- $\langle X \rangle$ denotes the free monoid generated by the set X.
- $\langle X \mid R \rangle$ denotes the monoid generated by X, subject to $R \subseteq \langle X \rangle \times \langle X \rangle$.
- From now on, usually $M := \langle X \mid R \rangle$, for arbitrary X and $R \subseteq \langle X \rangle \times \langle X \rangle$.
- Write $a =_M b$ if $a, b \in \langle X \rangle$ reduce to the same element of $M = \langle X \mid R \rangle$.

New Terminology

- $\mathsf{Z}_M(a) := \{b \in \langle X \rangle \mid a =_M b\}$ is the *set of factorizations of $a \in M$*.
- $\mathsf{L}_M(a) := \{|b| \mid b \in \mathsf{Z}_M(a)\}$ is the *length set of $a \in M$*.

Grandfathered Terminology

- $\mathcal{L}(M) := \{\mathsf{L}_M(a) \mid a \in \langle X \rangle\}$ is the *system of length sets of M*.
- $\Delta(L)$ denotes the set of all distances of $L \in \mathcal{L}(M)$.
- $\Delta(M) := \bigcup_{L \in \mathcal{L}(M)} \Delta(L)$.
- $\mathcal{U}_k(M) := \bigcup\{L \in \mathcal{L}(M) \mid k \in L\}$ is *union of length sets* containing $k \in \mathbb{N}$.
- $\rho_k(M) := \sup(\mathcal{U}_k(M)) \in \mathbb{N} \cup \{\infty\}$ is the *kth elasticity of M*, for all $k \in \mathbb{N}$ (provided that $\mathcal{U}_k(M) \neq \emptyset$).

Working With Generators

Proposition

Let $M := \langle X \mid (e, f)\rangle$, where $e, f \in \langle X\rangle$, and let $d := ||e| - |f|| \in \mathbb{N}$.

1. If $d = 0$, then M is *half-factorial* (i.e., $\mathsf{L}_M(a) = \{|a|\}$, for all $a \in \langle X\rangle$).
2. If $d \neq 0$, then $\Delta(M) = \{d\}$, and $\mathsf{L}_M(a)$ is an arithmetic progression with difference d, for all $a \in \langle X\rangle$.

Proof of (2)

- Suppose that $d \neq 0$, and let $a \in \langle X\rangle$.
- We may assume that $|a| = \min(\mathsf{L}_M(a))$.
- If $\mathsf{Z}_M(a) = \{a\}$, then $\mathsf{L}_M(a) = \{|a|\}$ is vacuously an arithmetic progression (with difference d).
- So suppose that $b \in \mathsf{Z}_M(a) \setminus \{a\}$.

Proof of (2) Continued

- Since $a =_M b$ and $a \neq b$, there exist $c_0, \ldots, c_m \in \langle X \rangle$ such that $c_0 = b$, $c_m = a$, and each c_i is connected to c_{i+1} via an elementary transition.
- That is, for each $i < m$, there exist $d_i, g_i, g_i', h_i \in \langle X \rangle$ satisfying $(g_i, g_i') \in \{(e, f), (f, e)\}$, $c_i = d_i g_i h_i$, and $c_{i+1} = d_i g_i' h_i$.
- Thus $||c_{i+1}| - |c_i|| = d$ for each i, and $\{d\} = \Delta(\mathsf{L}_M(a))$ $(= \Delta(M))$.
- Given that $|a| = \min(\mathsf{L}_M(a))$, and hence $|a| \leq |b|$, we have

$$\{|a|, |a| + d, |a| + 2d, \ldots, |b|\} \subseteq \mathsf{L}_M(a) \cap (|a| + d\mathbb{N}).$$

- Since $b \in \mathsf{Z}_M(a) \setminus \{a\}$ was arbitrary, $\mathsf{L}_M(a)$ is an arithmetic progression with difference d.

Atoms vs. Generators

- Unlike atom-based factorization indicators, the generator-based ones depend on the specific presentation.
- E.g., the trivial monoid $M = \{1\}$ is *BF* (i.e., $\mathsf{L}_M(a)$ is finite for all $a \in M$) with respect to the standard presentation $\langle \emptyset \rangle$, but not with respect to the presentation $\langle x \mid (1, x) \rangle$.

Proposition

Let $M := \langle X \mid R \rangle$, and let $\Lambda(Y) := \{uyv \mid y \in Y; u, v \in M^\times\}$ for all $Y \subseteq X$.

1. $\mathcal{A}(M) = \Lambda(\mathcal{A}(M) \cap X)$. In particular, if M is reduced, then $\mathcal{A}(M) \subseteq X$.
2. If $X \subseteq \mathcal{A}(M)$, equivalently $\Lambda(X) = \mathcal{A}(M)$, then M is atomic.
3. Suppose that M is reduced and X is *irredundant* (i.e., no proper subset of X generates M). Then M is atomic if and only if $\mathcal{A}(M) = X$.

Quarks and Other Factorization Bases

- Suppose that $\preceq$ is a preorder (i.e., reflexive transitive relation) on the (arbitrary) monoid M.
- $a \in M$ is a $\preceq$-*unit* if $a \preceq 1 \preceq a$.
- A $\preceq$-non-unit $a \in M$ is a $\preceq$-*atom* if $a \neq bc$ for all $\preceq$-non-units $b, c \in M$.
- A $\preceq$-non-unit $a \in M$ is $\preceq$-*irreducible* if $a \neq bc$ for all $\preceq$-non-units $b, c \in M$ satisfying $b \prec a$ and $c \prec a$.
- A $\preceq$-non-unit $a \in M$ is a $\preceq$-*quark* if for all $b \in M$, $b \prec a$ implies that b is a $\preceq$-unit.

Theorem (Tringali, 2022)

Let $\preceq$ be a preorder on M that satisfies the descending chain condition. Then every $\preceq$-non-unit of M factors into a product of $\preceq$-irreducible elements.

Theorem (Cossu/Tringali, 2024)

Let $\preceq$ be a preorder on M such that there are only finitely many $\preceq$-irreducible elements. Then unions of minimal length sets are all finite.

Generators as Quarks

- For all $a, b \in M = \langle X \mid R \rangle$ write

$$a \preceq b \iff \min(\mathsf{L}_M(a)) \leq \min(\mathsf{L}_M(b)).$$

- Then $\preceq$ is a preorder on M.
- Recall that $a \in M$ is a $\preceq$-*quark* if $b \prec a$ implies that $b \preceq 1 \preceq b$, for all $b \in M$, and it is not the case that $a \preceq 1 \preceq a$.
- So $a \in M = \langle X \mid R \rangle$ is a $\preceq$-quark if and only if $a =_M x$ for some $x \in X$.
- If $X = \{a \in M \mid a =_M x\}$, then X is the set of $\preceq$-quarks of M.

Example

Let $M := \langle x, y, z \mid (x, yz) \rangle$. Then $\min(\mathsf{L}_M(yz)) = |x| = 1$, and 1 is the only $a \in M$ with $\min(\mathsf{L}_M(a)) < 1$. So yz is a $\preceq$-quark but not a generator.

Generators as a Factorization Basis

- For a monoid M, let $X \subseteq M \setminus \{1\}$, and let N the submonoid of M generated by X.
- Let $\phi : \langle X \rangle \to N$ be the homomorphism induced by sending each element of X to the corresponding copy in N, and $\phi(1) = 1$.
- Then $N \cong \langle X \mid \ker(\phi) \rangle$.
- So we can formulate the factorization theory of N (and hence also M) in terms of any building blocks (e.g., primes, atoms, quarks, irreducible elements) using the generator setting, by letting X be the set of those building blocks in M.
- Restricting to the submonoid N is analogous to considering factorizations into atoms only for elements of M that can be written as products of atoms.
- Usually we choose a generating set that has nice properties, and then generators, atoms, quarks, and other such building blocks coincide.

Example

Questions

For a finitely-presented unit-cancellative monoid M with finite elasticities $\rho_k(M)$, must it be the case that the differences $\rho_k(M) - \rho_{k-1}(M)$ are bounded? Must such a monoid satisfy the Structure Theorem for Unions?

- Let $M := \langle u, v, x, y \mid (u^2, v^3), (xy, y^n x) \rangle$, where $n > 1$.
- *Claim 1: $x^k y =_M y^{n^k} x^k$ for all $k \in \mathbb{N}$.*
  $x^k y = x^{k-1}(xy) =_M x^{k-1}(y^n x) =_M \cdots =_M x^{k-2}(y^{n^2} x^2) =_M \cdots$.
- *Claim 2: $\rho_k(M) = n^{k-1} + k - 1$ for all $k \in \mathbb{N}^+$.*
- *Claim 3: The differences $\rho_k(M) - \rho_{k-1}(M)$ are unbounded for $k \geq 2$.*
  $\rho_k(M) - \rho_{k-1}(M) = n^{k-1} + k - 1 - (n^{k-2} + k - 2) = n^{k-2}(n-1) + 1$.
- *Claim 4: For all $k \geq 3$, there is a maximal $\mu_k(M) \in \mathbb{N}^+$ such that $\{\mu_k(M) - 1, \mu_k(M)\} \subseteq \mathcal{U}_k(M)$, and $\mu_k(M) \leq \rho_{k-2}(M) + 4$.*
- *Claim 5: M is finitely-presented, cancellative, and has finite elasticities, but does not satisfy the Structure Theorem for Unions.*

Normalizing/Duo Monoids and Rings

Definition

If $aM = Ma$ for all $a \in M$, then the monoid M is *normalizing* or *duo*.

Definition

A ring R is *left*, respectively, *right duo*, if every left, respectively right, ideal of R is a two-sided ideal.

Example

If D is a division ring and σ is an automorphism of D, then the skew power series ring $D[[x;\sigma]]$ is (left and right) duo. (Multiplication in this ring is induced by $xr = \sigma(r)x$, for all $r \in D$.)

Example (Marks, 2004)

Let R be a left or right self-injective von Neumann regular ring. Then R is left/right duo if and only if the power series ring $R[[x]]$ is left/right duo.

Normalizing Monoids

Theorem (Jespers and Okniński, 1999)

Every *completely integrally closed* monoid, whose quotient groups are finitely-generated torsion-free nilpotent groups, and which satisfies the ascending chain condition on right ideals, is normalizing.

Theorem (Geroldinger, 2013)

Let M be a cancellative monoid satisfying the *left and right Ore conditions*, and let $q(M)$ denote the *quotient group* of M. Then M is a normalizing *Krull* monoid if and only if M has a *divisor theory*.

Theorem (Cossu/Tringali, 2023)

Every *left normalizing* monoid (i.e., $aM \subseteq Ma$ for all $a \in M$), which is *locally finitely generated up to units*, satisfies the ascending chain condition on principal two-sided ideals.

Normalizing Monoids

Theorem

Let $M := \langle X \mid R\rangle$, where X is finite but nonempty, and $\{|a| - |b| \mid (a, b) \in R\}$ is finite. Suppose that M is normalizing and cancellative. Then M satisfies the Structure Theorem for Unions. If M is additionally BF (i.e., $\mathsf{L}_M(a)$ is finite for all $a \in \langle X\rangle$), then it satisfies the Strong Structure Theorem for Unions.

$M := \langle X \mid R\rangle$ satisfies the *Structure Theorem for Unions* if there exist $d, k^* \in \mathbb{N}^+$ and $m \in \mathbb{N}$, such that for all $k \geq k^*$,

$$(k + d\mathbb{Z}) \cap [\lambda_k(M) + m, \rho_k(M) - m] \subseteq \mathcal{U}_k(M) \subseteq k + d\mathbb{Z}.$$

(Here $\mathcal{U}_k(M) = \bigcup\{L \in \mathcal{L}(M) \mid k \in L\}$, $\rho_k(M) = \sup(\mathcal{U}_k(M))$, and $\lambda_k(M) = \min(\mathcal{U}_k(\mathcal{L}))$.)

M satisfies the *Strong Structure Theorem for Unions* if M satisfies the Structure Theorem for Unions, and there exist $n, k^* \in \mathbb{N}^+$ such that the following hold for all $m \in \mathbb{N}$ and $k \geq k^*$:

1. $(\rho_k(M) - \mathcal{U}_k(M)) \cap [0, m] = (\rho_{k+n}(M) - \mathcal{U}_{k+n}(M)) \cap [0, m]$,
2. $(\mathcal{U}_k(M) - \lambda_k(M)) \cap [0, m] = (\mathcal{U}_{k+n}(M) - \lambda_{k+n}(M)) \cap [0, m]$.

Proof Sketch

1. If M is not BF (and so $\rho_n(M) = \infty$ for some $n \in \mathbb{N}^+$), then M satisfies the Structure Theorem for Unions, by a standard argument. So we may assume that M is BF. Since X is finite, we may also assume that it is irredundant (i.e., no proper subset of X generates M as a monoid).
2. Let $x, y \in X$. Then there exists $z \in X$ such that $xy =_M zx$.
   - Since M is normalizing, $xy =_M ax$ for some $a \in \langle X \rangle$.
   - Since M is cancellative and X is irredundant, $a \neq_M 1$, and hence $a \neq 1$.
   - So we can write $a = bz$ for some $b \in \langle X \rangle$ and $z \in X$.
   - Then $xy =_M ax =_M bzx =_M bxc =_M xdc$ for some $c, d \in \langle X \rangle$, with $c \neq 1$.
   - So $y =_M dc$.
   - Suppose that $b \neq_M 1$. Then $d \neq 1$, and so $|dc| \geq 2$. Since M is BF, dc cannot contain any instances of y, as a word, and so $y =_M dc$ contradicts X being irredundant.
   - Hence $b =_M 1$, and so $xy =_M zx$.
3. Write $X = \{x_1, \ldots, x_n\}$, for some $n \in \mathbb{N}^+$. Then, by induction on word length, one can show that for each $a \in \langle X \rangle$, there is a unique $\nu(a) \in \langle X \rangle$ such that $a =_M \nu(a)$, $|a| = |\nu(a)|$, $\nu(a) = x_1^{m_1} \cdots x_n^{m_n}$, and $(m_1, \ldots, m_n)$ is minimal in the lexicographic ordering on $\mathbb{N}^n$ among such n-tuples.

Proof Sketch (Continued)

4. M has *accepted elasticity*. That is, either $\mathcal{L}(M) = \emptyset$, or $\rho(M) = \rho(L) < \infty$ for some $L \in \mathcal{L}(M)$, where

$$\rho(L) := \frac{\sup(L \cap \mathbb{N}^+)}{\min(L \cap \mathbb{N}^+)} \quad \text{and} \quad \rho(M) := \sup\{\rho(L) \mid L \in \mathcal{L}(M)\}.$$

- Let $S := \{(\nu(a), \nu(b)) \in \langle X \rangle \times \langle X \rangle \mid a =_M b\} \setminus \{(x_1^0 \cdots x_n^0, x_1^0 \cdots x_n^0)\}$, and define $f : S \to \mathbb{N}^{2n}$ via

$$f((x_1^{k_1} \cdots x_n^{k_n}, x_1^{m_1} \cdots x_n^{m_n})) := (k_1, \ldots, k_n, m_1, \ldots, m_n).$$

- By Dickson's Theorem, there are only finitely many minimal points in $f(S)$, relative to the product order on $\mathbb{N}^{2n}$.
- Let $T \subseteq S$ be the inverse image under f of the set of minimal points. Since f is injective, T is finite, and so we can choose $(\nu(a), \nu(b)) \in T$ such that $|\nu(a)|/|\nu(b)|$ is maximal.
- Then one can show that for all $(\nu(c), \nu(d)) \in S$,

$$\frac{|\nu(c)|}{|\nu(d)|} \leq \frac{|\nu(a)|}{|\nu(b)|}, \text{ and hence } \rho(M) = \frac{\sup(\mathsf{L}_M(a))}{\min(\mathsf{L}_M(a))} = \frac{|a|}{|b|} = \rho(L_M(a)).$$

Proof Sketch (Continued)

5. Since M has accepted elasticity, it satisfies the Strong Structure Theorem for Unions, by the following result.

Definition

Let $\mathcal{L}$ be a collection of subsets of $\mathbb{N}$. Then $\mathcal{L}$ is *primitive* if

$$\bigcup_{L \in \mathcal{L}} L \cap \mathbb{N}^+ \neq \emptyset \quad \text{and} \quad \gcd\left(\bigcup_{L \in \mathcal{L}} L \cap \mathbb{N}^+\right) = 1,$$

and $\mathcal{L}$ is *subadditive* if for all $L_1, L_2 \in \mathcal{L}$, there exists $L_3 \in \mathcal{L}$ such that $L_1 + L_2 \subseteq L_3$.

Theorem (Tringali, 2019)

Let $\mathcal{L}$ be a subadditive primitive collection of subsets of $\mathbb{N}$, which has accepted elasticity. Then $\mathcal{L}$ satisfies the Strong Structure Theorem for Unions (appropriately defined).

A Consequence

Proposition

The following hold for any finitely-generated normalizing unit-cancellative monoid M.

1. M is atomic.
2. If M is reduced, then the set of atoms $\mathcal{A}(M)$ is the unique (necessarily finite) irredundant generating set for M.

Corollary (Fan/Geroldinger/Kainrath/Tringali, 2017)

Every finitely-generated commutative cancellative monoid has accepted elasticity, in terms of atoms.

Example

- Let $M := \langle x, y, z \mid (xy, yzx) \rangle$. Then M is finitely-presented, cancellative, and BF, but does not have accepted elasticity (and is not normalizing).
- *Claim 1: M is cancellative.*
  This follows from a result of Adyan (1960).
- *Claim 2:* $\rho(\mathsf{L}_M(x^k y^l)) = 2 - \frac{1}{k+l}$ *for all* $k, l \in \mathbb{N}^+$.
  This follows from the observation that $\mathsf{L}_M(x^k y^l) = [k + l, 2(k + l) - 1]$.
- *Claim 3: For all* $n \in \mathbb{N}$, *among* $a \in \langle x, y, z \rangle$ *such that* $n = |a|$, $\rho(\mathsf{L}_M(a))$ *is maximal for any* $a = x^k y^l$ *with* $n = k + l$.
- *Claim 4: M is BF but does not have accepted elasticity.*
  By Claims 2 and 3, $\rho(L) < 2$ for all $L \in \mathcal{L}(M)$, but

$$\rho(M) = \sup_{n \in \mathbb{N}^+} \left(2 - \frac{1}{n} \right) = 2.$$

Other Conditions on Generators

Let $M := \langle X \mid R \rangle$, and

$$Y := \{x \in X \mid \exists a, b, c \in \langle X \rangle \text{ such that } (axb, c) \in R \text{ or } (c, axb) \in R\}.$$

Theorem

If M has accepted elasticity, and $X \neq Y$, then M is fully elastic (i.e., for every $q \in \mathbb{Q}$ satisfying $1 < q < \rho(M)$, there exists $L \in \mathcal{L}(M)$ such that $\rho(L) = q$).

Proposition

If Y is finite, then the following are equivalent.

1. There exists $a \in \langle X \rangle$ such that $\mathsf{Z}_M(a)$ is infinite.
2. There exists $a \in \langle X \rangle$ such that $\mathsf{L}_M(a)$ is infinite.
3. There exists $k \in \mathbb{N}$ such that $\mathcal{U}_k(M)$ and $\rho_k(M)$ are infinite.
4. There exists $k \in \mathbb{N}$ such that $\mathcal{U}_n(M)$ and $\rho_n(M)$ are infinite for all $n \geq k$.

Questions

Question

For which (finite) X and R does $\langle X \mid R \rangle$ satisfy the Structure Theorem for Unions?

Question

For which (finite) X and R does $\langle X \mid R \rangle$ have accepted elasticity?

Question

For which (finite) X and R is $\langle X \mid R \rangle$ fully elastic?

Thank you!

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